

Ising-model description of the $SU(2)_1$ quantum critical point in a dimerized two-leg spin-1/2 ladder

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Abstract

A nonperturbative analytical description of the $SU(2)_1$ quantum critical point in an explicitly dimerized two-leg spin-1/2 Heisenberg ladder is presented. It is shown that this criticality essentially coincides with that emerging in a weakly dimerized spin-1 chain with a small Haldane gap. The approach is based on the mapping onto an $SO(3)$ -symmetric model of three strongly coupled quantum Ising chains. This mapping is used to establish the correspondence between all physical fields of the spin ladder and those characterizing the $SU(2)_1$ criticality at the infrared fixed point.

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1 Introduction

Quantum many-particle systems in one space dimension, such as 1D interacting electrons, antiferromagnetic spin chains and ladders, exhibit universal properties in the low-energy limit. The field-theoretical description of these properties is based on an appropriately chosen conformally invariant (critical) theory which is defined in the high-energy, ultraviolet (UV) limit and then deformed by a number of perturbations consistent with the structure and symmetry of the underlying

microscopic model. While a single relevant perturbation (with scaling dimension $d < 2$) drives the model away from criticality towards a strong-coupling gapped phase in the infrared (IR) limit, the interplay between several relevant perturbations can lead to a quantum critical point where conformal invariance is restored, though with a central charge c_{IR} less than that at the unstable UV fixed point [1]. Understanding of such phase transitions in 1D models, where powerful nonperturbative techniques are available, is of great importance because asymptotically exact results obtained in one space dimension can be relevant to the issue of quantum critical points in higher-dimensional systems, most notably, in two-dimensional cuprate superconductors (see e.g. [2] and references therein).

An example of Abelian field theory displaying a quantum critical point was recently discussed by Delfino and Mussardo [3]. They considered the double-frequency sine-Gordon (DSG) model, which is a two-dimensional Gaussian model perturbed by two relevant vertex operators with the ratio of their scaling dimensions $d_1/d_2 = 4$, and showed that, upon fine tuning of its parameters, the model exhibits an Ising criticality with central charge $c_{\text{IR}} = 1/2$. In non-Abelian field theories, the existence of quantum critical points belonging to the universality class of $\text{SU}(2)_k$ Wess-Zumino-Novikov-Witten (WZNW) model and resulting from the interplay between several symmetry-preserving relevant perturbations was anticipated some time ago by Affleck and Haldane [4]. They argued that a massive phase of a translationally invariant spin-chain Hamiltonian can be driven to criticality by an external, parity-breaking perturbation, such as an explicit dimerization. Apparently, the explicitly dimerized even-leg spin-1/2 ladders, which in the absence of external perturbations are known to possess a fully gapped excitation spectrum (see for a review Ref. [5]), are excellent candidates for such a scenario. A nice example of this kind has been recently given by Martin-Delgado *et al* [6, 7] (see also [8]) who considered the standard J - J_\perp two-leg spin ladder ($J, J_\perp > 0$)

$$H_{\text{stand}} = J \sum_n \sum_{i=1,2} \mathbf{S}_{i,n} \cdot \mathbf{S}_{i,n+1} + J_\perp \sum_n \mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n} , \quad (1)$$

and modified it by making the constituent chains dimerized with a relative phase π : $J \rightarrow J_{1,2}(n) = J \pm (-1)^n J'$. At $J = \pm J' = \frac{1}{2} J_\perp$ the dimerized ladder transforms to a snake looking, translationally invariant $S=1/2$ Heisenberg chain which is critical and belongs to the $\text{SU}(2)_1$ WZNW universality class with central charge $c = 1$ [4]. Using the mapping onto a nonlinear sigma-model with a topological (θ) term, in Ref. [6] (see also [9]) it was argued that, at a given $J > 0$, there should exist a critical line $J_\perp = J_\perp(J')$ along which the model displays the IR properties of a single $S=1/2$ Heisenberg chain.

An analytical description of the critical points resulting from the competition between different relevant perturbations requires an entirely nonperturbative scheme able to correctly identify those degrees of freedom that remain massive and the low-energy ones that eventually become critical. Such a scheme has

been recently proposed in Ref. [10] to tackle the Ising transition in the DSG model. The approach was based on the mapping onto a model of two quantum Ising (QI) chains coupled by an Ashkin-Teller (AT) interaction and also by a magnetic-field type coupling $h\sigma_1^z\sigma_2^z$. In the limit where the amplitude h is considered as the largest energy scale in the problem, the “relative” Ising degrees of freedom described by $\tau^z = \sigma_1^z\sigma_2^z$ become locked, while the “total” degrees of freedom, described e.g. by σ_1^z , asymptotically decouple and can be tuned to criticality. Universality then implies that such separation between the fast and slow modes is valid close to the transition at arbitrary h . This approach made it possible to study the Ising criticality in a number of applications of the DSG model to physical systems [10, 11].

In this paper, we extend this strategy to provide a consistent nonperturbative description of the $SU(2)_1$ criticality in the π -dimerized two-leg spin ladder. Such a possibility naturally follows from the correspondence between two weakly coupled Heisenberg chains and an $O(3)\times Z_2$ symmetric theory of four noncritical Ising models with order parameters σ_i , developed in Refs. [12, 13]. The new ingredient introduced by the explicit relative dimerization is a perturbation $h\prod_{i=1}^4\sigma_i$ with $h\sim J'$. We stress, however, that the problem of dimerized ladders is not only different from the DSG model in that instead of two Ising models here we have four, but most importantly because in the adopted Ising description and all subsequent nonlocal (“change-of-basis”) transformations one has to maintain the unbroken spin rotational symmetry. For this reason, the $SU(2)_1$ quantum criticality reveals itself in the Ising-model description in a rather nontrivial way. The so far existing conclusion on the existence of the $c=1$ critical point in the $S=1/2$ staggered ladder [6] and in the related model of the explicitly dimerized $S=1$ chain [14, 15] was reached either by mapping onto an $O(3)$ nonlinear sigma model which, strictly speaking, is valid for large S only, or using a perturbative (renormalization group) approach combined with numerical methods. In other cases [8], the $c=1$ criticality of the dimerized ladder (1) was analyzed by the Abelian bosonization in the way suitable only for the XXZ-version of the model but not for the $SU(2)$ symmetric case where the explicit dimerization field is the most relevant perturbation in the problem. Treating our QI-chain model in the strong-coupling limit and applying a nonlocal duality transformation [16, 17], below we *derive* the effective Hamiltonian describing the low-energy sector of the dimerized spin ladder which represents a lattice spin-1/2 Heisenberg model with bond-alternating nearest-neighbor interactions. The $SU(2)_1$ criticality is reached when the external dimerization enforces the exchange couplings on even and odd bonds to coincide. This mapping is then used to establish the correspondence between the physical fields of the spin ladder and those characterizing the $SU(2)_1$ criticality at the IR fixed point.

The paper is organized as follows. In Sec. II, we review the field-theoretical approach to a weakly coupled two-leg Heisenberg ladder and show that the $SU(2)_1$ criticality emerging upon the explicit π -phase dimerization of the constituent

chains [6] is essentially the quantum critical point appearing in a weakly dimerized effective spin-1 chain [14, 15] with a small Haldane gap [18]. In Sec. III, we introduce an $SO(3)$ symmetric quantum AT model extended to include an interaction $h\sigma_1\sigma_2\sigma_3$. This model, which represents a lattice version of the field theory discussed in Sec. II, is treated in the large- h limit, and the effective low-energy Hamiltonian displaying the $SU(2)_1$ criticality is derived. In Sec. IV, we establish the correspondence between the physical fields of the spin ladder and those characterizing the $SU(2)_1$ criticality at the IR fixed point. We conclude with a summary and discussion. The paper contains two Appendices where we provide some details concerning the nonlocal duality transformation and the derivation of the effective Hamiltonian in the large- h limit.

2 Spin ladder in the continuum limit

As shown in Refs. [12, 13], at $J_\perp \ll J$ the original lattice model (1) can be mapped onto an $O(3) \times Z_2$ -symmetric theory of four massive Majorana fermions, or equivalently, four noncritical 2D Ising models. In the continuum limit, the effective Hamiltonian density

$$\begin{aligned} \mathcal{H}_M = & -\frac{iv_t}{2} \left(\vec{\xi}_R \cdot \partial_x \vec{\xi}_R - \vec{\xi}_L \cdot \partial_x \vec{\xi}_L \right) - im_t \vec{\xi}_R \cdot \vec{\xi}_L \\ & -\frac{iv_s}{2} \left(\xi_R^4 \partial_x \xi_R^4 - \xi_L^4 \partial_x \xi_L^4 \right) - im_s \xi_R^4 \xi_L^4 \\ & + \frac{1}{2} g_1 \left(\vec{\xi}_R \cdot \vec{\xi}_L \right)^2 + g_2 \left(\vec{\xi}_R \cdot \vec{\xi}_L \right) \left(\xi_R^4 \xi_L^4 \right) \end{aligned} \quad (2)$$

describes a degenerate triplet of Majorana fields, $\vec{\xi} = (\xi^1, \xi^2, \xi^3)$, and a singlet Majorana field, ξ^4 . The mass terms in (2) originate from the relevant part of the interchain interaction, $J_\perp \mathbf{n}_1 \cdot \mathbf{n}_2$, where \mathbf{n}_i is the staggered magnetization of the i -th chain: $(-1)^n \mathbf{S}_{i,n} \rightarrow \mathbf{n}_i(x)$. The triplet and singlet masses are given by $m_t = \lambda J_\perp$, $m_s = -3\lambda J_\perp$, where $\lambda > 0$ is a nonuniversal constant. The marginal part of the transverse exchange, $J_\perp \mathbf{J}_1 \cdot \mathbf{J}_2$, expressed in terms of the smooth components of the local spin densities (vector “currents” \mathbf{J}_i), gives rise to the four-fermion interaction in (2).

While the right and left components of the vector currents $\mathbf{J}_i = \mathbf{J}_{iR} + \mathbf{J}_{iL}$ admit a local representation in terms of Majorana bilinears,

$$\begin{aligned} \mathbf{I}_\nu &= \mathbf{J}_{1\nu} + \mathbf{J}_{2\nu} = -\frac{i}{2} \left(\vec{\xi}_\nu \wedge \vec{\xi}_\nu \right), \\ \mathbf{K}_\nu &= \mathbf{J}_{1\nu} - \mathbf{J}_{2\nu} = i\vec{\xi}_\nu \xi_\nu^4; \quad (\nu = R, L) \end{aligned} \quad (3)$$

this is not so for the staggered fields \mathbf{n}_i and ϵ_i , the latter being the dimerization operator defined as $(-1)^n \mathbf{S}_{i,n} \cdot \mathbf{S}_{i,n+1} \rightarrow \epsilon_i(x)$. However, these fields with scaling

dimension 1/2 can be expressed as products of the order and disorder parameters, σ_a and μ_a , of the related Ising models (see Refs. [12, 19] for more details):

$$\mathbf{n}^+ \sim (\mu_1\sigma_2\sigma_3\mu_4, \sigma_1\mu_2\sigma_3\mu_4, \sigma_1\sigma_2\mu_3\mu_4) , \quad (4)$$

$$\mathbf{n}^- \sim (\sigma_1\mu_2\mu_3\sigma_4, \mu_1\sigma_2\mu_3\sigma_4, \mu_1\mu_2\sigma_3\sigma_4) , \quad (5)$$

$$\epsilon^+ \sim \mu_1\mu_2\mu_3\mu_4 , \quad \epsilon^- \sim \sigma_1\sigma_2\sigma_3\sigma_4 , \quad (6)$$

where $\mathbf{n}^\pm = \mathbf{n}_1 \pm \mathbf{n}_2$, $\epsilon^\pm = \epsilon_1 \pm \epsilon_2$. Since the Majorana mass is related to the temperature of the associated Ising model via the relation $m \sim (T - T_c)/T_c$, the triplet of Ising models is disordered ($m_t > 0$) while the singlet Ising system is ordered ($m_s < 0$). As follows from Eqs. (4)–(6), this fact plays a crucial role in the behaviour of the correlation functions. In particular, the ground state of the system is parity symmetric ($\langle \epsilon^\pm \rangle = 0$), and the dynamical spin susceptibility $\chi''(q, \omega)$, calculated by Fourier transforming the asymptotics of the correlator $\langle \mathbf{n}^-(x, \tau) \cdot \mathbf{n}^-(0, 0) \rangle$, exhibits the existence of a coherent $S=1$ magnon peak at $\omega^2 = (\pi - q)^2 v_t^2 + m_t^2$. Thus, at low energies, the standard two-leg ladder represents a Haldane's disordered spin liquids with \mathbf{n}^- playing the role of the staggered magnetization of the effective $S=1$ chain. In fact, the model (2) can be thought as a continuum low-energy theory of a spin-1 chain with a small Haldane gap, represented by a triplet of Majorana fermions [20, 21], coupled to a noncritical Ising model:

$$\begin{aligned} \mathcal{H} = & \frac{\pi v_t}{2} (\mathbf{I}_R \cdot \mathbf{I}_R + \mathbf{I}_L \cdot \mathbf{I}_L) + m_t \text{Tr } \hat{\Phi} + g_1 \mathbf{I}_R \cdot \mathbf{I}_L \\ & - \frac{i v_s}{2} (\xi_R^4 \partial_x \xi_R^4 - \xi_L^4 \partial_x \xi_L^4) - i m_s \xi_R^4 \xi_L^4 \\ & + g_2 \mathbf{K}_R \cdot \mathbf{K}_L + h \sigma_4 \text{Tr } \hat{g} . \end{aligned} \quad (7)$$

The first term in the r.h.side of (7) is the Hamiltonian of the critical $\text{SU}(2)_2$ WZNW model describing universal properties of the $S=1$ chain at the exactly integrable, multicritical point [22], $\mathbf{I}_{R,L}$ are the level-2 chiral vector currents defined in (3), $\hat{\Phi}$ is a 3×3 matrix field, which is a primary field of the $\text{SU}(2)_2$ WZNW model with scaling dimension $d_1 = 1$, ($\text{Tr } \hat{\Phi} = -i \vec{\xi}_R \cdot \vec{\xi}_L$), and \hat{g} is the WZNW field in the fundamental (2×2) representation, with dimension $d_2 = 3/8$. The operator

$$\epsilon_t = \text{Tr } \hat{g} = \sigma_1 \sigma_2 \sigma_3 \quad (8)$$

is a parity-breaking, dimerization field of the triplet sector which couples to the order parameter σ_4 of the singlet Ising system. Since the latter is in the ordered phase, in the lowest order one can replace in Eq. (7) the operators σ_4 and $\xi_R^4 \xi_L^4$ by their expectation values and only deal with the triplet sector. Thus, the original problem of the dimerized spin ladder reduces to the problem of a spin-1 chain with a small Haldane gap and a weak bond alternation:

$$\mathcal{H}_t = \frac{\pi v_t}{2} (\mathbf{I}_R \cdot \mathbf{I}_R + \mathbf{I}_L \cdot \mathbf{I}_L) + \tilde{g}_1 \mathbf{I}_R \cdot \mathbf{I}_L + \tilde{m}_t \text{Tr } \hat{\Phi} + \bar{h} \epsilon_t , \quad (9)$$

where \tilde{g}_1 and \tilde{m}_t are renormalized values of the coupling constant and triplet mass, respectively, and

$$\bar{h} = h\langle\sigma_4\rangle . \quad (10)$$

The neglected residual interaction between the singlet modes and those low-energy triplet degrees of freedom that eventually become critical affects the precise location of the $c = 1$ critical point but does not alter the universal properties of the transition.

3 Generalized SO(3) quantum Ashkin-Teller model and the SU(2)₁ criticality

Guided by the well-known correspondence between the theory of a massive Majorana fermion and a QI model, i.e. the Ising chain in a transverse magnetic field [23, 24], in this section we consider the following model of three coupled QI chains

$$\begin{aligned} H_{\text{QI}}^{(3)} = & - \sum_{a=1,2,3} \sum_n \left(\mathcal{J} \sigma_{a,n}^z \sigma_{a,n+1}^z + \Delta \sigma_{a,n}^x \right) \\ & - K \sum_{a < b} \sum_n \left(\sigma_{a,n}^z \sigma_{a,n+1}^z \sigma_{b,n}^z \sigma_{b,n+1}^z + \sigma_{a,n}^x \sigma_{b,n}^x \right) \\ & + \bar{h} \sum_n \sigma_{1,n}^z \sigma_{2,n}^z \sigma_{3,n}^z . \end{aligned} \quad (11)$$

The AT interaction between the chains is chosen to be self-dual and parametrized by a single constant K . The last term in (11) couples all three chains together.

Let us first check that the lattice Hamiltonian (11) is indeed SO(3)-symmetric and then show that, in the continuum limit, it reduces to the model (9). The hidden SO(3) symmetry of the Hamiltonian (11) becomes manifest in the Majorana representation [25]. We introduce lattice Majorana fields, $\eta_{a,n}$ and $\zeta_{a,n}$ ($a = 1, 2, 3$), satisfying the anticommutation relations

$$\begin{aligned} \{\eta_{a,n}, \eta_{b,m}\} &= \{\zeta_{a,n}, \zeta_{b,m}\} = 2\delta_{ab}\delta_{nm} , \\ \{\eta_{a,n}, \zeta_{b,m}\} &= 0 , \end{aligned} \quad (12)$$

and then make use of the Jordan-Wigner transformations:

$$\sigma_{a,n}^x = i\zeta_{a,n}\eta_{a,n} , \quad \sigma_{a,n}^z = i\kappa_a \left(\prod_{m=1}^n \sigma_{a,m}^x \right) \zeta_{a,n} . \quad (13)$$

Here κ_a are Klein factors which anticommute among themselves,

$$\{\kappa_a, \kappa_b\} = 2\delta_{ab} , \quad (14)$$

but commute with all the other operators. Quantum disorder operators $\mu_{a,n+1/2}^{x,z}$, defined on the dual lattice $\{n + 1/2\}$, are given by

$$\mu_{a,n+1/2}^x = -i\zeta_{a,n}\eta_{a,n+1} , \quad \mu_{a,n+1/2}^z = \prod_{m=1}^n \sigma_{a,m}^x . \quad (15)$$

The order and disorder operators are related to each other by the duality transformations:

$$\sigma_{a,n}^z \sigma_{a,n+1}^z = \mu_{a,n+1/2}^x, \quad \mu_{a,n-1/2}^z \mu_{a,n+1/2}^z = \sigma_{a,n}^x. \quad (16)$$

With the definitions (13), (15), the lattice Majorana fields

$$\eta_{a,n} = \kappa_a \sigma_{a,n}^z \mu_{a,n-1/2}^z = \kappa_a \mu_{a,n-1/2}^z \sigma_{a,n}^z, \quad (17)$$

$$\zeta_{a,n} = i\kappa_a \sigma_{a,n}^z \mu_{a,n+1/2}^z = -i\kappa_a \mu_{a,n+1/2}^z \sigma_{a,n}^z, \quad (18)$$

satisfy the required anticommutation relations (12).

Using Eqs. (13), we fermionize the Hamiltonian (11). The first line in (11) immediately transforms to an O(3)-symmetric sum of three lattice Majorana models:

$$H_0 = i \sum_n \left(\vec{\mathcal{J}}_{\zeta_n} \cdot \vec{\eta}_{n+1} - \Delta \vec{\zeta}_n \cdot \vec{\eta}_n \right).$$

The AT part of $H_{\text{QI}}^{(3)}$ transforms to

$$H_{\text{AT}} = K \sum_{a < b} \sum_n [(\zeta_{a,n} \eta_{a,n+1}) (\zeta_{b,n} \eta_{b,n+1}) + (\zeta_{a,n} \eta_{a,n}) (\zeta_{b,n} \eta_{b,n})], \quad (19)$$

and is also O(3)-symmetric because

$$\sum_{a < b} (\zeta_{a,n} \eta_{a,m}) (\zeta_{b,n} \eta_{b,m}) = \frac{3}{2} + \frac{1}{2} (\vec{\zeta}_n \cdot \vec{\eta}_m)^2.$$

The product of $\sigma_{1,n}^z \sigma_{2,n}^z \sigma_{3,n}^z$ involves products of three Majorana fields, $\eta_{1m} \eta_{2m} \eta_{3m}$ and $\zeta_{1m} \zeta_{2m} \zeta_{3m}$. Notice that the product of the three components of a Majorana triplet, $\eta_1 \eta_2 \eta_3 = (1/3!) \epsilon^{abc} \eta_a \eta_b \eta_c$, is an SO(3) invariant object. Therefore the \bar{h} -perturbation in (11) only breaks the discrete subgroup $Z_2 \subset \text{O}(3)$, thus reducing the symmetry of the model (11) to SO(3).

If the triplet of QI chains is close to the critical point and the interchain interaction is weak,

$$|\Delta - \mathcal{J}| \ll \mathcal{J} \quad \text{and} \quad K, h \ll \mathcal{J}, \quad (20)$$

one can pass to the continuum limit in which the lattice operators $\eta_{a,n}$ and $\zeta_{a,n}$ are replaced by slowly varying Majorana fields

$$\eta_{a,n} \rightarrow \sqrt{2a_0} \eta_a(x), \quad \zeta_{a,n} \rightarrow \sqrt{2a_0} \zeta_a(x).$$

Here the factor $\sqrt{2}$ ensures the correct continuum anticommutation relations, $\{\eta_i(x), \eta_j(y)\} = \{\zeta_i(x), \zeta_j(y)\} = \delta_{ij} \delta(x - y)$. The Hamiltonian density of three decoupled QI chains then becomes

$$\mathcal{H}_0(x) \rightarrow [iv \vec{\zeta}(x) \cdot \partial_x \vec{\eta}(x) - im_t \vec{\zeta}(x) \cdot \vec{\eta}(x)] \quad (21)$$

with $v = 2\mathcal{J}a_0$ and $m_t = 2(\Delta - \mathcal{J})$. A global chiral rotation of the Majorana spinors,

$$\xi_{aR} = \frac{-\eta_a + \zeta_a}{\sqrt{2}}, \quad \xi_{aL} = \frac{\eta_a + \zeta_a}{\sqrt{2}}, \quad (22)$$

transforms (21) to the noninteracting triplet part of the Hamiltonian (2). On the other hand, up to irrelevant corrections, H_{AT} in (19) transforms to the marginal interaction terms in (2) with $g = 8Ka_0$. The correspondence between the \bar{h} -terms in (11) and (9) is self-evident.

Thus, we have shown that, in the weak-coupling limit (20), the 1D quantum model (11) can be regarded as a symmetry preserving lattice counterpart of the continuum theory (9). General universality considerations allow us to expect that, if the field-theoretical model (9) displays a certain quantum critical behavior, this should also be a property of the quantum lattice model (11) even when its parameters are not restricted by the condition (20). It is then legitimate to consider the model (11) in the strong-coupling, large- \bar{h} , limit where the description of the $\text{SU}(2)_1$ criticality greatly simplifies.

Let us pass to a new set of Ising variables, $s_{1,n}^z, s_{2,n}^z, \tau_n^z$, where

$$s_{1,n}^z = \sigma_{1,n}^z, \quad s_{2,n}^z = \sigma_{2,n}^z, \quad \tau_n^z = \sigma_{1,n}^z \sigma_{2,n}^z \sigma_{3,n}^z. \quad (23)$$

In the original $(\sigma_1, \sigma_2, \sigma_3)$ representation, the local (at a given site n) Hilbert space of the three-chain model is spanned on the basis vectors $|\sigma_1, \sigma_2, \sigma_3\rangle$ which are eigenstates of σ_1^z, σ_2^z and σ_3^z :

$$\sigma_a^z |\sigma_1, \sigma_2, \sigma_3\rangle = \sigma_a |\sigma_1, \sigma_2, \sigma_3\rangle, \quad \sigma_a = \pm 1. \quad (a = 1, 2, 3)$$

The new local basis $|s_1, s_2, \tau\rangle$ is defined as

$$\begin{aligned} s_a^z |s_1, s_2, \tau\rangle &= s_a |s_1, s_2, \tau\rangle, \quad (a = 1, 2) \\ \tau^z |s_1, s_2, \tau\rangle &= \tau |s_1, s_2, \tau\rangle, \end{aligned}$$

where $s_a = \sigma_a$, $\tau = \sigma_1 \sigma_2 \sigma_3$. Comparing matrix elements of the operators $\sigma_{a,n}^\alpha$ ($a = 1, 2, 3$) in the two bases, we find the following correspondence:

$$\begin{aligned} \sigma_{1,n}^z &= s_{1,n}^z, \quad \sigma_{2,n}^z = s_{2,n}^z, \quad \sigma_{3,n}^z = s_{1,n}^z s_{2,n}^z \tau_n^z, \\ \sigma_{1,n}^x &= s_{1,n}^x \tau_n^x, \quad \sigma_{2,n}^x = s_{2,n}^x \tau_n^x, \quad \sigma_{3,n}^x = \tau_n^x. \end{aligned} \quad (24)$$

Let $\nu_{1,n+1/2}^\alpha, \nu_{2,n+1/2}^\alpha, \rho_{n+1/2}^\alpha$ be the disorder operators dual to $s_{1,n}^\alpha, s_{2,n}^\alpha, \tau_n^\alpha$, respectively, and obeying the duality relations similar to (16). Under the change of basis, the original dual spins $\mu_{a,n+1/2}^\alpha$ ($a = 1, 2, 3$) transform as follows:

$$\begin{aligned} \mu_{1,n+1/2}^z &= \nu_{1,n+1/2}^z \rho_{n+1/2}^z, \quad \mu_{2,n+1/2}^z = \nu_{2,n+1/2}^z \rho_{n+1/2}^z, \\ \mu_{3,n+1/2}^z &= \rho_{n+1/2}^z, \\ \mu_{1,n+1/2}^x &= \nu_{1,n+1/2}^x, \quad \mu_{2,n+1/2}^x = \nu_{2,n+1/2}^x, \\ \mu_{3,n+1/2}^x &= \nu_{1,n+1/2}^x \nu_{2,n+1/2}^x \rho_{n+1/2}^x. \end{aligned} \quad (25)$$

It is easy to check that the new pairs of mutually dual operators, (s_1, ν_1) , (s_2, ν_2) and (τ, ρ) , satisfy the same algebra, Eqs. (17), (18), as the original operators (σ_a, μ_a) ($a = 1, 2, 3$).

In terms of the new variables, the Hamiltonian (11) reads:

$$\begin{aligned}
H_{\text{QI}}^{(3)} = & -(\mathcal{J} + K) \sum_n \left(s_{1,n}^z s_{1,n+1}^z + s_{2,n}^z s_{2,n+1}^z + s_{1,n}^z s_{1,n+1}^z s_{2,n}^z s_{2,n+1}^z \right) \\
& -K \sum_n \left(s_{1,n}^x + s_{2,n}^x + s_{1,n}^x s_{2,n}^x \right) \\
& + \bar{h} \sum_n \tau_n^z - \Delta \sum_n \left(s_{1,n}^x + s_{2,n}^x + 1 \right) \tau_n^x \\
& + \frac{\mathcal{J}}{2} \sum_n s_{1,n}^z s_{1,n+1}^z s_{2,n}^z s_{2,n+1}^z \left(\tau_n^z - \tau_{n+1}^z \right)^2 \\
& + \frac{K}{2} \sum_n \left(s_{1,n}^z s_{1,n+1}^z + s_{2,n}^z s_{2,n+1}^z \right) \left(\tau_n^z - \tau_{n+1}^z \right)^2 .
\end{aligned} \tag{26}$$

The advantage of this representation of $H_{\text{QI}}^{(3)}$ becomes transparent in the strong-coupling limit:

$$\bar{h} \gg \mathcal{J}, \Delta, K . \tag{27}$$

In the zeroth-order approximation, the Hamiltonian (26) describes a collection of noninteracting τ -spins in a strong magnetic field, with a large energy gap $2\bar{h}$ separating the fully polarized ground state from the excited states with one spin flip. Thus the τ degrees of freedom are “fast” and can therefore be integrated out to produce an effective Hamiltonian for the low-energy, (s_1, s_2) part of the spectrum.

Using a unitary transformation in the form of a $1/\bar{h}$ expansion, we project $H_{\text{QI}}^{(3)}$ onto the lowest-energy state ($\tau^n = -1, \forall n$) of the zeroth-order Hamiltonian, $H_0 = \bar{h} \sum_n \tau_n^z$, and thus obtain the effective model of two coupled QI chains, $H_{\text{eff}}[s_1, s_2]$. Apparently, the symmetry of $H_{\text{eff}}[s_1, s_2]$ might appear only as $Z_2 \times Z_2$, for $H_{\text{QI}}^{(3)}$ in (26) contains only $s_{1,2}^x$ and $s_{1,2}^z$. The hidden $SU(2)$ symmetry of the effective model is revealed by the nonlocal duality transformation first introduced by Kohmoto, den Nijs and Kadanoff (KNK) in their study of the 2D AT model [16], and later on employed by Kohmoto and Tasaki in their analysis of the bond-alternating $S=1/2$ chain [17]. This transformation, which is briefly reviewed in Appendix A, establishes an important correspondence between two combinations of the variables s_1^α, s_2^α ($\alpha = z, x$) and local $SU(2)$ invariants written in terms of new spin-1/2 operators \mathbf{T}_n :

$$s_{1,n}^z s_{1,n+1}^z + s_{2,n}^z s_{2,n+1}^z + s_{1,n}^z s_{1,n+1}^z s_{2,n}^z s_{2,n+1}^z \rightarrow -4 \mathbf{T}_{2,n} \cdot \mathbf{T}_{2n+1} , \tag{28}$$

$$s_{1,n}^x + s_{2,n}^x + s_{1,n}^x s_{2,n}^x \rightarrow -4 \mathbf{T}_{2n-1} \cdot \mathbf{T}_{2,n} , \tag{29}$$

where $\{2n\}$ and $\{2n+1\}$ denote even and odd sublattices of a new lattice with $2N$ sites. Notice that the duality transformation that maps the l.h.sides of Eqs. (28)

and (29) onto each other is equivalent to a translation by one lattice spacing on the lattice where the spin-1/2 operators \mathbf{T}_n are defined.

To the accuracy $O(1/\bar{h}^2)$, the low-energy effective Hamiltonian is found to be (see Appendix B)

$$H_{\text{eff}}[s_1, s_2] = \sum_{n=1}^N (J_1 \mathbf{T}_{2,n} \cdot \mathbf{T}_{2n+1} + J_2 \mathbf{T}_{2n-1} \cdot \mathbf{T}_{2,n}) + J_3 \sum_{n=1}^N (\mathbf{T}_{2n-1} \cdot \mathbf{T}_{2n+1} + \mathbf{T}_{2,n} \cdot \mathbf{T}_{2n+2}) , \quad (30)$$

where

$$J_1 = 4 \left[\mathcal{J} + K - \left(\frac{\Delta}{\bar{h}} \right)^2 (\mathcal{J} + 2K) \right] , \\ J_2 = 4 \left(K + \frac{\Delta^2}{\bar{h}} \right) , \quad J_3 = 4 \left(\frac{\Delta}{\bar{h}} \right)^2 K . \quad (31)$$

$H_{\text{eff}}[s_1, s_2]$ is manifestly $\text{SU}(2)$ invariant and describes a spin-1/2 chain with bond-alternating nearest-neighbor interactions J_1 and J_2 and the next-nearest-neighbor interaction J_3 . The $\text{SU}(2)_1$ critical regime occurs when the bond alternation vanishes: $J_1 = J_2$. Notice that the condition $\Delta/\bar{h} \ll 1$ ensures that the frustrating interaction J_3 remains irrelevant at the transition. Eqs. (30), (31) constitute the central result of our work.

It is worth stressing here that the equivalence between two coupled critical QI chains and a bond-alternating $\text{SU}(2)$ -symmetric $S=1/2$ Heisenberg model necessarily implies that the AT interchain interaction is of the order of the cutoff and fine tuned in such a way that all three terms in the l.h.sides of Eqs. (28), (29) have the same amplitude. This is a manifestation of the well-known fact, implicitly present in earlier studies of the 2D AT model [16], that there exists no free-fermion Majorana representation of the critical $\text{SU}(2)_1$ WZNW model. At the point where the effective $N=2$ AT model $H_{\text{eff}}[s_1, s_2]$ is self-dual, the $S=1/2$ Heisenberg chain becomes translationally invariant and critical. In the 2D $N=2$ AT model, this is the point where the $c=1$ line of Gaussian critical points with continuously varying exponents bifurcates into two Ising critical lines [16, 26].

In the lowest order, the critical value of \bar{h} is given by $\bar{h}_c = \Delta^2/\mathcal{J}$. With Δ/\bar{h} being the small parameter in the expansions (31), the ratio $\Delta/\mathcal{J} \sim \bar{h}_c/\Delta \gg 1$, implying that the $c=1$ criticality is reached if the three QI chains in (11) are strongly disordered. This is a direct consequence of the imposed condition (27) which prevents us to establish a quantitative correspondence between the parameters of the field-theoretical models (7), (9) and the lattice model (11), and, in particular, determine the critical line. However, universality arguments allow us to expect that the asymptotic decoupling between the τ -degrees of freedom, which become frozen at the $\text{SU}(2)_1$ transition, and the (s_1, s_2) degrees of freedom

that describe an effective critical $S=1/2$ Heisenberg chain is a general property of the quantum AT model (11) with an arbitrary value of \bar{h} . This means that, even when the condition (27) is released, the low-energy sector of the model (11) will still be described by an effective Hamiltonian (30), although the determination of its parameters will remain as a complicated, yet unresolved, part of the problem.

Despite this shortcoming, it is possible to extract the scaling of the critical line in the weak-coupling limit from general considerations and compare the results with the already existing ones. Let us go back to the model (9). At small \bar{h} and \tilde{m}_t , the scaling law for the critical line $\bar{h}_c = \bar{h}_c(\tilde{m}_t)$ simply follows from the comparison of the mass gaps, $\sim \bar{h}^{8/13}$ and $\sim \tilde{m}_t$, generated by the \bar{h} - and \tilde{m}_t -perturbations independently [15]:

$$\bar{h}_c \sim [\tilde{m}_t]^{13/8} \sim J_\perp^{13/8} . \quad (32)$$

For the standard ladder where $|m_s| \simeq 3|m_t| \sim J_\perp$, one has $\langle \sigma_4^z \rangle \sim J_\perp^{1/8}$, and the relation (32) translates to

$$h_c \sim \langle \sigma_4^z \rangle^{-1} J_\perp^{13/8} \sim J_\perp^{3/2} , \quad (33)$$

in agreement with the result of Ref. [7].

Returning to the adopted mean-field treatment of the singlet Ising component, it is worth noticing here that the description of the lowest-energy part of the spectrum in terms of the effective spin-1/2 Hamiltonian (30) is valid not only at the $SU(2)_1$ transition but also at small deviations from criticality, provided that the associated (dimerization) mass gap $m_{\text{dimer}} \sim |\bar{h} - \bar{h}_c|^{2/3}$ is much smaller than the singlet mass $|m_s|$.

4 UV-IR transmutation of physical fields

Let us now find out how the physical fields, characterizing the spin ladder in the UV limit, are transformed at the IR fixed point where the system becomes $SU(2)_1$ critical.

We start with the current operators $\mathbf{J}_{1,2}$ which, according to Eqs. (3), are expressed in terms of the Majorana fields $\vec{\xi}$ and ξ^4 . Let us first consider the total current,

$$\mathbf{I} = \mathbf{J}_1 + \mathbf{J}_2 = -\frac{i}{2} \left(\vec{\xi}_R \wedge \vec{\xi}_R + \vec{\xi}_L \wedge \vec{\xi}_L \right) ,$$

which is fully determined in the triplet sector of the model and, in the UV limit, actually represents the level-2 vector current of the $SU(2)_2$ WZNW model given by the first term in Eq. (9). This current is nothing but the smooth part of the magnetization of the related critical $S=1$ Heisenberg chain. The x -component of this current is

$$I^x(x) = -i \left[\xi_R^2(x) \xi_R^3(x) + \xi_L^2(x) \xi_L^3(x) \right] . \quad (34)$$

Making the chiral rotation (22), we can define a local lattice operator

$$I_n^x = -\frac{i}{2} (\eta_{2,n} \eta_{3,n} + \zeta_{2,n} \zeta_{3,n}) \rightarrow a_0 I^x(x) , \quad (35)$$

which transforms back to (34) in the continuum limit. Using definitions (17), (18), together with the “change-of-basis” transformations (24), (25), we obtain:

$$\begin{aligned} I_n^x &= \frac{i\kappa_2\kappa_3}{2} \sigma_{2,n}^z \sigma_{3,n}^z \left(\mu_{2,n+1/2}^z \mu_{3,n+1/2}^z - \mu_{2,n-1/2}^z \mu_{3,n-1/2}^z \right) \\ &= \frac{i\kappa_2\kappa_3}{2} \tau_n^z s_{1,n}^z \left(\nu_{2,n+1/2}^z - \nu_{2,n-1/2}^z \right) . \end{aligned} \quad (36)$$

Since τ_n^z is a noncritical field, it can be replaced by its nonzero expectation value. Using formulas (61) of Appendix A, we express I_n^x in terms of the spin operators of the effective critical $S=1/2$ Heisenberg chain (30):

$$I_n^x = i\kappa_2\kappa_3\kappa_1 \langle \tau^z \rangle \left(T_{2n}^x + T_{2n+1}^x \right) . \quad (37)$$

In the continuum limit

$$\begin{aligned} \mathbf{T}_m &\rightarrow (a_0/2) \mathbf{T}(x) , \\ \mathbf{T}(x) &= \mathbf{J}(x) + (-1)^m \mathbf{n}(x) , \end{aligned} \quad (38)$$

where $\mathbf{J} = \mathbf{J}_R + \mathbf{J}_L$ and \mathbf{n} are the smooth and staggered parts of the local magnetization $\mathbf{T}(x)$. In (38) we took into account the fact that, by the KNK construction (see Appendix A), the lattice spacing of the effective $S=1/2$ Heisenberg chain is $a_0/2$. Keeping only relevant operators, we find that $I^x(x)$, being the level-2 current in the UV limit, transforms to

$$I^x(x) \rightarrow i\kappa_2\kappa_3\kappa_1 \langle \tau^z \rangle [J^x(x) - (a_0/2) \partial_x n^x(x) + \dots] , \quad (39)$$

where \dots stand for irrelevant corrections. Quite similarly we find that

$$I^y(x) \rightarrow i\kappa_3\kappa_1\kappa_2 \langle \tau^z \rangle [J^y(x) - (a_0/2) \partial_x n^y(x) + \dots] , \quad (40)$$

$$I^z(x) \rightarrow [J^z(x) - (a_0/2) \partial_x n^z(x) + \dots] . \quad (41)$$

Since $(\kappa_1\kappa_2\kappa_3)^2 = -1$, we can replace $\kappa_1\kappa_2\kappa_3$ by i . In the leading order in $1/\hbar$, $\langle \tau^z \rangle = -1$, and we arrive at the following correspondence

$$\mathbf{I}(x) \rightarrow \mathbf{J}(x) - C a_0 \partial_x \mathbf{n}(x) , \quad (42)$$

(C being a nonuniversal constant), which shows that the level-2 vector current \mathbf{I} , originally defined at the UV fixed point, transforms in the IR limit not only to the level-1 vector current \mathbf{J} but also acquires a nonholomorphic piece represented by the operator $\partial_x \mathbf{n}$ with scaling dimension $3/2$ and conformal spin 1.

One can similarly show that the level-2 axial vector current (i.e. the spin current of the critical S=1 chain), defined at the UV fixed point as

$$\mathbf{I}_5 = -\frac{i}{2} \left(\vec{\xi}_R \wedge \vec{\xi}_R - \vec{\xi}_L \wedge \vec{\xi}_L \right),$$

transforms to

$$\mathbf{I}_5(x) \rightarrow \mathbf{J}_5(x) + C a_0 \partial_t \mathbf{n}(x) \quad (43)$$

where $\mathbf{J}_5 = \mathbf{J}_R - \mathbf{J}_L$ is the axial vector current of the $SU(2)_1$ WZNW model (i.e. the spin current of the S=1/2 Heisenberg chains). Notice that the constant C in (43) is the same as in (42). As a result, both at the UV and IR fixed points, the currents \mathbf{I} and \mathbf{I}_5 satisfy the continuity equation:

$$\partial_t \mathbf{I} + \partial_x \mathbf{I}_5 = 0.$$

Since the holomorphic property of both level-2 currents is lost in the IR limit, the second equation $\partial_x \mathbf{I} + \partial_t \mathbf{I}_5 = 0$ is no longer valid.

According to Eq. (3), the relative current $\mathbf{K} = \mathbf{J}_1 - \mathbf{J}_2$ involves the singlet Majorana field ξ^4 which remains massive across the transition. Therefore, the current \mathbf{K} represents a short-ranged field whose correlations decay exponentially over the length scale $\xi_s \sim v_s/|m_s|$.

Let us now turn to the staggered fields \mathbf{n}^\pm and ϵ^\pm which are defined at the UV fixed point by Eqs. (4)–(6). Since \mathbf{n}^+ and ϵ^+ are proportional to μ_4 and since the forth Ising model is ordered ($m_s < 0$, $\langle \mu_4 \rangle = 0$), these two fields remain short-ranged at the transition. On the other hand, with σ_4 replaced by its nonzero expectation value, \mathbf{n}^- and ϵ^- transform to the staggered magnetization and dimerization field of the effective $S=1$ chain

$$\mathbf{n}^- \sim (\sigma_1 \mu_2 \mu_3, \mu_1 \sigma_2 \mu_3, \mu_1 \mu_2 \sigma_3), \quad (44)$$

$$\epsilon^- \sim \epsilon_t = \sigma_1 \sigma_2 \sigma_3. \quad (45)$$

Adopting a symmetric lattice definition, for $(n^-)^x$ we obtain:

$$(n^-)_n^x = \frac{1}{2} \sigma_{1,n} \left(\mu_{2,n+1/2}^z \mu_{3,n+1/2}^z + \mu_{2,n-1/2}^z \mu_{3,n-1/2}^z \right). \quad (46)$$

Under transformations (24), (25) and formulas (61), we find that $(n^-)_n^x$ becomes

$$\begin{aligned} (n^-)_n^x &= \frac{1}{2} s_{1,n}^z \left(\nu_{2,n+1/2}^z - \nu_{2,n-1/2}^z \right) \\ &= \frac{1}{2} \kappa_1 \left(T_{2,n}^x - T_{2,n-1}^x \right). \end{aligned}$$

Passing then to the continuum limit, with (38) taken into account, we obtain:

$$(n^-)^x(x) \sim \kappa_1 n_x, \quad (47)$$

where \mathbf{n} is the staggered magnetization of the critical $S=1/2$ chain. Similarly

$$(n^-)^y(x) \sim \kappa_2 n_y, \quad (n^-)^z(x) \sim i\kappa_1 \kappa_2 \langle \tau^z \rangle n_x. \quad (48)$$

The Klein factors can be identified with Pauli matrices:

$$\kappa_1 = \tilde{\tau}^x, \quad \kappa_2 = \tilde{\tau}^y, \quad i\kappa_1 \kappa_2 = -\tilde{\tau}^z.$$

Thus, in the leading order in $1/\bar{h}$ ($\langle \tau^z \rangle = -1$), we arrive at the following correspondence:

$$(n^-)^\alpha(x) \sim \tilde{\tau}^\alpha n^\alpha. \quad (\alpha = x, y, z) \quad (49)$$

The presence of the Klein factors in (49) should not be confusing. These factors drop out of the Hamiltonians (11) and (30) and have no dynamics. Moreover, due to the unbroken $SU(2)$ symmetry, $\langle n^\alpha(x, \tau) n^\beta(0, 0) \rangle = \frac{1}{3} \delta^{\alpha\beta} \langle \mathbf{n}(x, \tau) \cdot \mathbf{n}(0, 0) \rangle$, the Klein factors drop out of the correlation functions as well. Thus, at the $SU(2)_1$ critical point, the relative staggered magnetization of the spin ladder transforms to the staggered magnetization of the effective $S=1/2$ chain:

$$\mathbf{n}^- \rightarrow \mathbf{n}. \quad (50)$$

It may seem from formulas (39), (40) and (48) that taking into account $1/\bar{h}$ corrections to $\langle \tau^z \rangle = -1$ would break $SU(2)$ invariance of the UV-IR transmutation rules (42), (50). The point is that the adopted “minimal” choice of local lattice operators (see Eqs. (35), (46)), although being consistent with the continuum representation of the corresponding fields, is not unique. Other lattice regularizations of the field operators may be equally good in this respect but differ in the \bar{h} -dependence of the prefactors and subleading corrections at the critical point. With the manifestly $SU(2)$ invariant correspondence (42), (50) firmly established in the leading order, we can only state here that these corrections are not universal and, as already explained in the preceding section, cannot be addressed within this approach.

Finally, we consider transmutation of the dimerization field ϵ^- . From (45) and (24) it follows that

$$\epsilon^- \sim \tau^z \sim -I. \quad (51)$$

However, ϵ^- couples to the amplitude \bar{h} of the external dimerization whose variation leads to the deviation from criticality. The leading $SU(2)$ -symmetric, parity-breaking, relevant perturbation at the $SU(2)_1$ criticality is the dimerization field of the effective $S=1/2$ Heisenberg chain: $(-1)^n \mathbf{T}_n \cdot \mathbf{T}_{n+1} \rightarrow \epsilon(x)$. Therefore, the improved version of the formula (51), which includes a strongly fluctuating correction to the identity operator I , should read

$$\epsilon^-(x) \sim -I + \epsilon(x). \quad (52)$$

Since the scaling dimension of ϵ is $1/2$, the behavior of $\langle \epsilon \rangle$ close to the critical point is given by

$$|\langle \epsilon^- \rangle|_{\bar{h}} = |\langle \epsilon^- \rangle|_{\bar{h}_c} + \text{const.} |\bar{h} - \bar{h}_c|^{1/3} \text{sign}(\bar{h} - \bar{h}_c). \quad (53)$$

Thus, the average dimerization remains finite at the transition but has an infinite slope.

Let us now draw a physical picture emerging from the obtained results. In the limit of a translationally invariant ladder, $J'/J_\perp \rightarrow 0$, the lowest-energy part of the spin fluctuation spectrum displays a single coherent $S=1$ massive magnon formed due to confinement of the originally massless $S=1/2$ spinons of individual chains. In the opposite limit of two decoupled bond-alternating spin-1/2 chains, $J_\perp/J' \rightarrow 0$, coherent massive magnons are still present in the spectrum but are of a different nature: these represent two independent sets of triplet states formed by a soliton, antisoliton and the first breather of the two corresponding $\beta^2 = 2\pi$ sine-Gordon models (see e.g. Ref. [19], chapter 22). The crossover between these two extreme cases involves a critical point where the spectrum is entirely incoherent and consists of pairs of $S=1/2$ spinons of the effective $S=1/2$ Heisenberg chain. (The behavior of the current-current correlation function $\langle \mathbf{I}(x, \tau) \mathbf{I}(0, 0) \rangle$ at the transition will be modified due to the admixture of the nonholomorphic operator in Eq. (42)). As follows from (30), (31), close to the transition, the low-energy properties of the system are those of a *single*, weakly dimerized spin-1/2 Heisenberg chain. (The fact that the average relative dimerization of the original ladder, $\langle \epsilon^- \rangle$, stays finite across the transition implies that higher-energy degrees of freedom which remain massive at the critical point are also dimerized.)

The two massive phases separated by the $c = 1$ critical point differ in the sign of the explicit dimerization of the effective $S=1/2$ chain (30) and are characterized by two different string order parameters [27], each of them being nonzero only in one phase and vanishing in the other. For the spin-1/2 alternating Heisenberg chain, the nonlocal string order parameter associated with the breakdown of a hidden $Z_2 \times Z_2$ symmetry was first considered by Kohmoto and Tasaki [17] and Hida [28]. Its representation in terms of the Ising order and disorder parameters for two-leg spin-1/2 ladders was introduced in Refs. [12, 13]. The two different string order parameters of the alternating chain (30) are defined as follows (due to the unbroken $SU(2)$ symmetry, it is sufficient to consider only the x -components of the corresponding operators):

$$\begin{aligned} O_{2k, 2n-1}^x &= \exp \left(i\pi \sum_{j=2k}^{2n-1} T_j^x \right) , \\ O_{2k+1, 2n}^x &= \exp \left(i\pi \sum_{j=2k+1}^{2n} T_j^x \right) . \end{aligned} \quad (54)$$

Using relations (62) and (63) given in Appendix A, we find that

$$O_{2k, 2n-1}^x = s_{1,k}^z s_{1,n}^z , \quad O_{2k+1, 2n}^x = \mu_{2,k+1/2}^z \mu_{2,n+1/2}^z , \quad (55)$$

where $(s_{1,k}^z, \mu_{1,k+1/2}^z)$ and $(s_{2,k}^z, \mu_{2,k+1/2}^z)$ are pairs of the order and disorder operators of two QI chains representing the effective spin-1/2 Heisenberg Hamiltonian

(30) (see Sec. III). As follows from (31), close to the transition, $J_1 - J_2 \sim \bar{h} - \bar{h}_c$. Since at $\bar{h} < \bar{h}_c$ the two QI chains are disordered, one finds that

$$\langle O_{2k+1,2n}^x \rangle \neq 0, \quad \langle O_{2k,2n-1}^x \rangle = 0. \quad (56)$$

At $\bar{h} > \bar{h}_c$ these chains are ordered implying that in this case

$$\langle O_{2k+1,2n}^x \rangle = 0, \quad \langle O_{2k,2n-1}^x \rangle \neq 0. \quad (57)$$

As shown by Hida [28], at $\bar{h} - \bar{h}_c \rightarrow 0$, the two string order parameters vanish as $|\bar{h} - \bar{h}_c|^{1/6}$.

5 Summary and discussion

In this paper, we have proposed a nonperturbative approach to describe the $SU(2)_1$ criticality in the dimerized, weakly-coupled two-leg spin-1/2 ladder. We have shown that this criticality is in fact a quantum critical point of the effective spin-1 Haldane spin liquid perturbed by the explicit dimerization. Using the mapping onto a generalized, $SO(3)$ -symmetric, quantum Ashkin-Teller model and employing a nonlocal duality transformation, we have derived the low-energy effective Hamiltonian which represents a lattice $S=1/2$ Heisenberg chain with a small bond alternation. The $SU(2)_1$ critical point corresponds to the case when fine tuning of the parameters of the model restores translational invariance. With the adopted approach we were able to find an asymptotically exact correspondence between the physical fields of the original spin ladder and those characterizing the $SU(2)_1$ criticality at the IR fixed point.

To avoid confusion, let us emphasize that the effective spin-1/2 Heisenberg model (30) should not be misleadingly associated with the one that corresponds to a *strongly* dimerized, snake looking two-leg ladder. The Hamiltonian $H_{\text{eff}}[s_1, s_2]$ was derived in the large- h limit from the generalized, $SO(3)$ -symmetric, quantum AT model (11) which should be regarded as a regularized lattice version of a continuum field theory (9), the latter describing universal properties of a *weakly* dimerized spin-1 chain. No such field-theoretical description is available if the in-chain staggering amplitude J' of the original spin ladder is not small.

The Ising-model description of quantum critical points in two-leg spin-1/2 ladders can be extended to a more interesting situation where all four Ising models associated with the triplet and singlet Majorana modes (see Eqs. (2) and (7)) become equally important. This turns out to be the case for a generalized spin ladder [13]

$$H_{\text{gen}} = H_{\text{stand}} + V \sum_n (\mathbf{S}_{1,n} \cdot \mathbf{S}_{1,n+1}) (\mathbf{S}_{2,n} \cdot \mathbf{S}_{2,n+1}), \quad (58)$$

which, apart from the standard on-rung exchange J_{\perp} present in H , Eq. (1), also includes a biquadratic interchain interaction. At $J_{\perp} = 0$ the model (58) is known

as the spin-orbital chain[29] and has been recently studied by different groups (see e.g. [30, 8, 31]). If $J_\perp \neq 0$ but $|V|$ is large enough, the model (58) occurs in a non-Haldane, spontaneously dimerized phase where the spectrum is entirely incoherent and consists of pairs of topological kinks[13]. When an external longitudinal dimerization of the chains is included, the generalized ladder is expected to exhibit a pattern of quantum criticalities presumably richer than that in the standard-ladder case discussed in Ref. [6] and in the present paper. The underlying $\text{SO}(3) \times \text{Z}_2$ symmetry of the model (58) opens a room for criticalities with central charge $c = 1/2, 1$ and $3/2$, corresponding to the universality classes of the Ising model and $\text{SU}(2)_k$ WZNW models with $k = 1$ and 2 . Recent numerical results[32] strongly suggest the appearance of critical points due to the interplay between the biquadratic interaction V and the explicit dimerization. This and related problems are presently under investigation.

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A KNK duality transformation

Consider a model of two noncritical QI chains with an AT-like interchain interaction

$$H_{\text{QI}} = - \sum_{a=1,2} \sum_{j=1}^N \left(A_a s_{a,j}^z s_{a,j+1}^z + B_a s_{a,j}^x \right) - \sum_{j=1}^N \left(A_3 s_{1,j}^z s_{1,j+1}^z s_{2,j}^z s_{2,j+1}^z + B_3 s_{1,j}^x s_{2,j}^x \right). \quad (59)$$

The KNK transformation [16, 17], which is a special combination of spin rotation and duality transformation, provides a mapping of the model (59) onto a quantum spin-1/2 chain. Here we outline basic steps of this transformation:

(i) duality transformation in the second QI copy:

$$s_{2,j}^x = \nu_{2,j-1/2}^z \nu_{2,j+1/2}^z, \quad s_{2,j}^z s_{2,j+1}^z = \nu_{2,j+1/2}^x;$$

(ii) reduction of the original ($\{j\}$) and dual ($\{j + 1/2\}$) chains to a single chain via the following identification:

$$s_{1,j}^\alpha \Rightarrow \mu_{2j}^\alpha, \quad \nu_{2,j+1/2}^\alpha \Rightarrow \mu_{2j+1}^\alpha;$$

(iii) relabeling the lattice sites, $j \rightarrow j - 1/4$, and treating the sublattice $\{2j + 1/2\}$ as dual to $\{2j\}$.

(iv) duality transformation:

$$\sigma_n^x = \mu_{n-1/2}^z \mu_{n+1/2}^z, \quad \sigma_n^z \sigma_{n+1}^z = \mu_{n+1/2}^x;$$

(v) a staggered π -rotation around the y -axis:

$$\sigma_n^z \rightarrow (-1)^n \sigma_n^z, \quad \sigma_n^x \rightarrow (-1)^n \sigma_n^x;$$

(vi) global $-\pi/2$ -rotation around the x -axis:

$$\sigma_n^y \rightarrow -\sigma_n^z, \quad \sigma_n^z \rightarrow \sigma_n^y.$$

This brings the Hamiltonian (59) to its final form — a bond-alternating XYZ $S=1/2$ chain:

$$H_{XYZ} = 4 \sum_{j=1}^N \sum_{\alpha=x,y,z} \left(J_1^\alpha T_{2j}^\alpha T_{2j+1}^\alpha + J_2^\alpha T_{2j-1}^\alpha T_{2j}^\alpha \right), \quad (60)$$

where $T_i^\alpha = (1/2)\sigma_i^\alpha$ are the spin-1/2 operators, and

$$\begin{aligned} J_1^x &= A_1, & J_1^y &= A_2, & J_1^z &= A_3; \\ J_2^x &= B_2, & J_2^y &= B_1, & J_2^z &= B_3. \end{aligned}$$

The KNK transformation establishes a correspondence between the spin operators on the even and odd lattice sites, T_{2j}^α and T_{2j+1}^α , and products of two Ising order and disorder operators of the two-chain model (59), (s_1, ν_1) and (s_2, ν_2) :

$$\begin{aligned} 2T_{2j}^x &= \kappa_1 s_{1,j}^z \nu_{2,j+1/2}^z, & 2T_{2j+1}^x &= -\kappa_1 s_{1,j+1}^z \nu_{2,j+1/2}^z, \\ 2T_{2j}^y &= \kappa_2 \nu_{1,j+1/2}^z s_{2,j}^z, & 2T_{2j+1}^y &= -\kappa_2 \nu_{1,j+1/2}^z s_{2,j+1}^z, \\ 2T_{2j}^z &= i(\kappa_1 \kappa_2) (s_{1,j}^z s_{2,j}^z) (\nu_{1,j+1/2}^z \nu_{2,j+1/2}^z), \\ 2T_{2j+1}^z &= -i(\kappa_1 \kappa_2) (s_{1,j+1}^z s_{2,j+1}^z) (\nu_{1,j+1/2}^z \nu_{2,j+1/2}^z). \end{aligned} \quad (61)$$

Given the standard algebra of the operators $s_{1(2),j}^z$ and $\nu_{1(2),j+1/2}$ (see Eqs. (17), (18)), the Klein factors κ_1, κ_2 in (61) ensure the correct algebra of the Pauli matrices $2\mathbf{T}_n$.

From (61) it follows that

$$4T_{2j}^x T_{2j+1}^x = -s_{1,j}^z s_{1,j+1}^z, \quad 4T_{2j}^y T_{2j+1}^y = -s_{2,j}^z s_{2,j+1}^z, \quad (62)$$

$$4T_{2j}^z T_{2j+1}^z = -s_{1,j}^z s_{1,j+1}^z s_{2,j}^z s_{2,j+1}^z;$$

$$4T_{2j-1}^x T_{2j}^x = -s_{2,j}^x, \quad 4T_{2j-1}^y T_{2j}^y = -s_{1,j}^x, \quad (63)$$

$$4T_{2j-1}^z T_{2j}^z = -s_{1,j}^x s_{2,j}^x.$$

Eqs. (62), (63) lead to the correspondence (29). Therefore, the hidden $SU(2)$ symmetry of the following model of two coupled QI chains

$$\begin{aligned} H &= - \sum_{j=1}^N [B(s_{1,j}^x + s_{2,j}^x + s_{1,j}^x s_{2,j}^x) \\ &\quad + A(s_{1,j}^z s_{1,j+1}^z + s_{2,j}^z s_{2,j+1}^z + s_{1,j}^z s_{1,j+1}^z s_{2,j}^z s_{2,j+1}^z)] \end{aligned} \quad (64)$$

is encoded in the special Ising structure of the A and B terms in the r.h.side of (64).

B Strong-coupling expansion

In this Appendix, we provide some details concerning the unitary transformation of the original model (26) which makes it possible to derive the effective Hamiltonian $H_{\text{eff}}[s_1, s_2]$ in the form of $1/\hbar$ expansion.

It is suitable to rearrange the Hamiltonian in (26) in the following way:

$$\begin{aligned} H_{\text{QI}} &= H + V, \\ H &= H_0 + W, \quad W = W_1 + W_2, \end{aligned} \quad (65)$$

where

$$\begin{aligned} H_0 &= \hbar \sum_n \tau_n^z, \quad V = -\Delta \sum_n Q_n(s) \tau_n^x, \\ W_1 &= -(\mathcal{J} + K) \sum_n \Lambda_{n,n+1}(s) - K \sum_n R_n(s), \\ W_2 &= \frac{1}{2} \sum_n P_{n,n+1}(s) (\tau_n^z - \tau_{n+1}^z)^2. \end{aligned} \quad (66)$$

In Eqs. (66)

$$\begin{aligned} Q_n(s) &= s_{1,n}^x + s_{2,n}^x + 1, \\ R_n(s) &= s_{1,n}^x + s_{2,n}^x + s_{1,n}^x s_{2,n}^x, \\ \Lambda_{n,n+1}(s) &= s_{1,n}^z s_{1,n+1}^z + s_{2,n}^z s_{2,n+1}^z + s_{1,n}^z s_{1,n+1}^z s_{2,n}^z s_{2,n+1}^z, \\ P_{n,n+1}(s) &= \mathcal{J} s_{1,n}^z s_{1,n+1}^z s_{2,n}^z s_{2,n+1}^z + K (s_{1,n}^z s_{1,n+1}^z + s_{2,n}^z s_{2,n+1}^z). \end{aligned} \quad (67)$$

Suppose that the eigenvalue problem for H is solved:

$$H|a\rangle = E_a|a\rangle.$$

Since $[H_0, W] = 0$, any state $|a\rangle$ can be represented as a direct product $|a\rangle = |\tau\rangle \otimes |s\rangle$, where $|s\rangle$ only involve quantum numbers characterizing the (s_1, s_2) part of the spectrum, while

$$|\tau\rangle = \prod_i^N |\tau_i\rangle$$

with $|\tau_i\rangle = |\uparrow\rangle_i, |\downarrow\rangle_i$ being eigenstates of the operator τ_i^z . Under the condition (27), the spectrum of H coincides in the leading order with that of H_0 . The ground state of H_0 is a fully polarized state

$$|0\rangle_\tau = |\downarrow\downarrow\downarrow \dots \downarrow\rangle_\tau, \quad (68)$$

while the lowest excited states involve one τ -spin flip

$$|\uparrow\downarrow\downarrow\downarrow\cdots\downarrow\rangle_\tau, |\downarrow\uparrow\downarrow\downarrow\cdots\downarrow\rangle_\tau, |\downarrow\downarrow\uparrow\downarrow\cdots\downarrow\rangle_\tau\cdots, \quad (69)$$

and have a gap $2\bar{h}$. With the s -degrees of freedom taken into account ($H_0 \rightarrow H_0 + V$), the ground state and 1-flip excited states transform to narrow bands. Our goal is to eliminate V in the lowest order and project the resulting Hamiltonian onto the state $|0\rangle_\tau$.

The procedure is standard. Consider a unitary transformation

$$\begin{aligned} \tilde{H}_{\text{QI}} &= e^S H_{\text{QI}} e^{-S} \\ &= H + V + [S, H] + [S, V] + \frac{1}{2} [S, [S, H]] + \cdots, \end{aligned} \quad (70)$$

where $S = -S^\dagger$. Requiring that $V + [S, H] = 0$, one finds that, in the lowest order in Δ/\bar{h} , the projected Hamiltonian

$$P_0 \tilde{H}_{\text{QI}} P_0 = -2N\bar{h} + W_1 + \frac{1}{2} P_0 [S, V] P_0 + O\left[(\Delta/\bar{h})^2\right], \quad (71)$$

where

$$P_0 = |0\rangle_\tau \langle 0|_\tau.$$

The matrix element of the commutator in (71) is given by

$$\begin{aligned} \langle a | [S, V] | b \rangle &= - \sum_c V_{ac} V_{cb} \left(\frac{1}{E_c - E_a} + \frac{1}{E_c - E_b} \right) \\ &= - \int_0^\infty d\lambda e^{-2\hbar\lambda} \sum_c \left[\tilde{V}_{ac}(\lambda) V_{cb} + V_{ac} \tilde{V}_{cb}(-\lambda) \right], \end{aligned} \quad (72)$$

where

$$\tilde{V}(\lambda) = e^{\lambda W} V e^{-\lambda W}.$$

In obtaining (72), we assumed that the states $|a\rangle$ and $|b\rangle$ are of the form $|0\rangle_\tau \otimes |s\rangle$. We also took into account the fact that the energy differences in (72) are necessarily positive (of the order of $2\bar{h}$) because the off-diagonal operator V connects $|0\rangle_\tau$ with 1-flip states (69). Using $1/\bar{h}$ expansion, we obtain:

$$P_0 [S, V] P_0 = -P_0 \left(\frac{1}{\bar{h}} V^2 + \frac{1}{4\bar{h}^2} [[W, V], V] \right) P_0 + O\left(\frac{1}{\bar{h}^3}\right). \quad (73)$$

Calculating the commutators in (73) and using formulas (62) and (63), one arrives at the results (30), (31) given in the main text.

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